

MODELS OF PA: STANDARD SYTEMS WITHOUT MINIMAL ULTRAFILTERS

SAHARON SHELAH

ABSTRACT. We prove that \mathbb{N} , the standard model of arithmetic, has an uncountable elementary extension N such that there is no ultrafilter on the Boolean Algebra of subsets of \mathbb{N} represented in N which is minimal (i.e. as in Rudin-Keisler order for partitions represented in N).

1. INTRODUCTION

Enayat [1], Question III, asked (see Definition 1.4(1)):

Question 1.1. Can we prove in ZFC that there is an arithmetically closed $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that \mathcal{A} carries no minimal ultrafilter?

He proved the existence of examples, for the stronger notion “2-Ramsey ultrafilter”. In [9] we prove that there is an arithmetically closed Borel set $\mathbf{B} \subseteq \mathcal{P}(\mathbb{N})$ such that any expansion \mathbb{N}^+ of \mathbb{N} by any uncountably many members of \mathbf{B} has this property, i.e. the family of definable subsets of \mathbb{N}^+ carries no 2-Ramsey ultrafilter.

We deal here with Question 1.1, proving that there is such a family of cardinality \aleph_1 , this implies the version in the abstract; (since it is well-known that every arithmetically closed family of cardinality at most \aleph_1 can be realized as the standard system of some elementary extension of \mathbb{N} , as shown by Knight and Nadel [3]). We use forcing but the result is proved in ZFC. On other problems from [1] see Enayat-Shelah [8] and [7], [9].

Notation 1.2. 1) Let $\text{pr}:\omega \times \omega \rightarrow \omega$ be the standard pairing function (i.e. $\text{pr}(n, m) = \binom{n+m}{2} + n$, so one to one onto two-place function).

2) Let \mathcal{A} denote a subset of $\mathcal{P}(\omega)$.

3) Let $\text{BA}(\mathcal{A})$ be the Boolean algebra which $\mathcal{A} \cup \{\omega\}^{<\aleph_0}$ generates.

4) Let D denote a non-principal ultrafilter on \mathcal{A} , meaning that $D \subseteq \mathcal{A}$ and there is a unique non-principal ultrafilter D' on the Boolean algebra $\text{BA}(\mathcal{A})$ satisfying

1991 *Mathematics Subject Classification.* Primary 03C62; Secondary: 03C50, 03C55, 03E40.

Key words and phrases. model theory, set theoretic model theory, Peano arithmetic, forcing, minimal ultrafilter.

$D = D' \cap \mathcal{A}$, notice that in Definition 1.4 below the distinction between an ultrafilter on \mathcal{A} and on $\text{BA}(\mathcal{A})$ makes a difference.

- 5) τ denotes a vocabulary extending $\tau_{\text{PA}} = \tau_{\mathbb{N}} = \{0, 1, +, \times, <\}$, usually countable.
- 6) $\text{PA}(\tau)$ is Peano arithmetic for the vocabulary τ . A model N of $\text{PA}(\tau)$ is called ordinary if $N \upharpoonright \tau_{\text{PA}}$ extends \mathbb{N} ; usually the models will be ordinary.
- 7) $\varphi(N, \bar{a})$ is $\{b : N \models \varphi[b, \bar{a}]\}$ where $\varphi(x, \bar{y}) \in \mathbb{L}(\tau_N)$ and $\bar{a} \in {}^{\ell_{g(\bar{y})}}N$.
- 8) $\text{Sym}(A)$ is the set (or group) of permutations of N .
- 9) For sets u, v of ordinals let $\text{OP}_{v,u}$, “the order preserving function from u to v ” be defined by: $\text{OP}_{v,u}(\alpha) = \beta$ iff $\beta \in v, \alpha \in u$ and $\text{otp}(v \cap \beta) = \text{otp}(u \cap \alpha)$.
- 10) We say $u, v \subseteq \text{Ord}$ form a Δ -system pair when $\text{otp}(u) = \text{otp}(v)$ and $\text{OP}_{v,u}$ is the identity on $u \cap v$.

Definition 1.3. 1) For $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\text{ar-cl}(\mathcal{A}) = \{B \subseteq \omega : B \text{ is first order defined in } (\mathbb{N}, A_1, \dots, A_n) \text{ for some } n < \omega \text{ and } A_1, \dots, A_n \in \mathcal{A}\}$. This is called the arithmetic closure of \mathcal{A} .

2) For a model N of $\text{PA}(\tau)$ let the standard system of N , $\text{SSy}(N)$ be $\{\varphi(M, \bar{a}) \cap \mathbb{N} : \varphi(x, \bar{y}) \in \mathbb{L}(\tau) \text{ and } \bar{a} \in {}^{\ell_{g(\bar{y})}}M\}$ so $\subseteq \mathcal{P}(\omega)$ for any ordinary model M isomorphic to N , see 1.2(6).

Definition 1.4. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

0) Let $\text{cd}_0 : \mathcal{H}(\aleph_0) \rightarrow \omega$ be one to one, and interpreting $\mathcal{H}(\aleph_0)$ inside \mathbb{N} it is (first order) definable by a bounded formula in \mathbb{N} , i.e. $\{\text{cd}_0(x, y) : x \in y \in \mathcal{H}(\aleph_0)\}$ is, and it maps $\mathbb{N} \times \mathbb{N}$ into \mathbb{N} . For $h \in {}^\omega \omega$ let $\text{cd}(h) = \{\text{pr}(n, h(n)) : n < \omega\}$, where pr is the standard pairing function of ω , see 1.2(1) and generally for $H \subseteq \mathcal{H}(\aleph_0)$ we let $\text{cd}(H) := \{\text{cd}_0(x) : x \in H\}$; this applies, e.g. to $h \in {}^{[\omega]^k} \omega$.

1) D , an ultrafilter on \mathcal{A} , is called minimal when: if $h \in {}^\omega \omega$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h \upharpoonright X$ is constant or one-to-one.

2) D , an ultrafilter on \mathcal{A} , is called Ramsey when: if $k < \omega$ and $h : [\omega]^k \rightarrow \{0, 1\}$ and $\text{cd}(h) \in \mathcal{A}$ then for some $X \in D$ we have $h \upharpoonright [X]^k$ is constant. Similarly k -Ramsey.

3) D , a non-principal ultrafilter on \mathcal{A} is called a Q-point when if $h \in {}^\omega \omega$ is increasing and $\text{cd}(h) \in \mathcal{A}$ then for some increasing sequence $\langle n_i : i < \omega \rangle$ we have $i < \omega \Rightarrow h(2i) \leq n_i < h(2i+1)$ and $\{n_i : i < \omega\} \in D$.

Remark 1.5. In [9] we also use the following notions:

1) D is called 2.5-Ramsey or self-definably closed when: if $\bar{h} = \langle h_i : i < \omega \rangle$ and $h_i \in {}^\omega(i+1)$ and $\text{cd}(\bar{h}) = \{\text{cd}(i, \text{cd}(n, h_i(n)) : i < \omega, n < \omega\}$ belongs to \mathcal{A} then for some $g \in {}^\omega \omega$ we have: $\text{cd}(g) \in \mathcal{A}$ and $(\forall i)[g(i) \leq i \wedge \{n < \omega : h_i(n) = g(i)\} \in D]$; this follows from 3-Ramsey and implies 2-Ramsey.

2) D is weakly definably closed when: if $\langle A_i : i < \omega \rangle$ is a sequence of subsets of ω and $\{\text{pr}(n, i) : n \in A_i \text{ and } i < \omega\} \in \mathcal{A}$, then $\{i : A_i \in D\} \in D$ (follows from 2-Ramsey).

Definition 1.6. 1) $\mathbb{L}(\mathbf{Q})$ is first order logic when we add the quantifier \mathbf{Q} where $(\mathbf{Q}x)\varphi$ means that there are uncountable many x 's satisfying φ .

2) $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ is defined parallelly.

See on those logics Keisler [2]. We shall use Laver forcing in the proof of Theorem 2.1, so let us define this forcing notion.

Definition 1.7. Let $T \subseteq \omega^{>\omega}$ be a subtree. For $a \in T$ let

$$\text{suc}_T(a) = \{a \hat{\ } \langle i \rangle \in T : i \in \omega\}.$$

The trunk $\text{tr}(T)$ of T is a maximal element $a \in T$ such that $a \leq_T b$ or $b \leq_T a$ for every $b \in T$. Such a tree T will be called a Laver tree iff $s = \text{tr}(T)$ and for every $t \in T$ such that $s \leq t$, the set $\text{suc}_T(t)$ is infinite.

We define the forcing notion \mathbb{Q} (= Laver forcing) as follows.

A condition $T \in \mathbb{Q}$ is a Laver tree. If $S, T \in \mathbb{Q}$ then $S \leq_{\mathbb{Q}} T$ iff $S \supseteq T$. If $\mathbf{G} \subseteq \mathbb{Q}$ is generic, then $\eta[\mathbf{G}] := \{a \in \omega^{>\omega} : \exists T \in \mathbf{G}, a \text{ is the trunk of } T\}$ will be called a Laver real.

Claim 1.8. If \boxtimes then \boxplus where:

- \boxtimes (a) $\bar{\mathbb{Q}} = \langle \mathbb{P}_{\alpha}, \mathbb{Q}_{\beta} : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$ is a CS iteration;
- (b) $k(*) < \omega$ and $\beta(k) < \alpha(*) < \omega_1$ for $k < k(*)$;
- (c) each \mathbb{Q}_{α} is a Laver forcing (in $\mathbf{V}^{\mathbb{P}_{\alpha}}$) and η_{α} its generic;
- (d) $h \in (\omega^{\omega})^{\mathbf{V}}$;
- (e) $p \in \mathbb{P}_{\alpha(*)}$;
- (f) $p \Vdash_{\mathbb{P}_{\alpha(*)}}$ “ $\underline{B}_k \subseteq \omega$ and $|\underline{B}_k \cap [\eta_{\beta(k)}(n+1), \eta_{\beta(k)}(n+2)]| \leq h(\eta_{\beta(k)}(n))$, for every n large enough” for $k < k(*)$;
- \boxplus for some p_1, p_2 and B_k^* , for $k < k(*)$, we have:
 - (a) $\mathbb{P}_{\alpha(*)} \Vdash$ “ $p \leq p_{\ell}$ ” for $\ell = 1, 2$;
 - (b) $B_k^* \subseteq \omega$ (from \mathbf{V});
 - (c) $p_1 \Vdash$ “ $\underline{B}_k \subseteq^* B_k^*$ ”;
 - (d) $p_2 \Vdash$ “ $\underline{B}_k \subseteq^* (\omega \setminus B_k^*)$ ”.

Proof. 1.8. Clearly letting $\underline{B}_* = \cup \{B_k : k < k(*)\}$, we have

- (*) $p \Vdash_{\mathbb{P}_{\alpha(*)}}$ “for every large enough n , the set $\underline{B}_* \cap [\eta_0(n+1), \eta_0(n+2)]$ has $\leq \eta_0(n)$ members”.

Now by the properties of iterating Laver forcing ([4] or see [5, Ch. VI]), we have:

- (*) if $\mathbf{G}_1 \subseteq \mathbb{P}_1$ is generic over \mathbf{V} and $\eta = \eta_0[\mathbf{G}_1]$ then

$\Vdash_{\mathbb{P}_{\alpha(*)}/\mathbf{G}_1}$ “if $\underline{B} \subseteq \omega$ and in $\underline{B} \cap [\eta(n), \eta(n+1))$ there are $\leq \eta(n)$ elements for every n large enough then for some $B' \in \mathbf{V}[\mathbf{G}_1]$, $B' \subseteq \omega$, $\underline{B} \subseteq B'$ and $B' \cap [\eta(n), \eta(n+1))$ has $\leq (\eta(n))^n$ members for every n large enough”.

Now this applies in particular to $\underline{B} = \underline{B}_*$ getting \underline{B}' . Hence without loss of generality $\alpha(*) = 1$ so we can replace \mathbb{P}_1 by \mathbb{Q}_0 , Laver forcing; also for a dense set of

$p \in \mathbb{Q}_0$ we have: if $\eta \in p$ is of length $n+1$ so an increasing sequence of natural numbers, then $p^{[\eta]} := \{v \in p : v \triangleleft \eta \text{ or } \eta \triangleleft v\}$ forces a value b_η to $\mathcal{B}' \cap [0, \eta(n))$ so necessarily $|b_\eta| \leq \eta(n-1)$ when $n > 1$.

By thinning p , without loss of generality if $\eta \in p$ and $u_\eta = \{n : \eta \hat{\ } \langle n \rangle \in p\}$ is infinite (equivalently is not a singleton) then $\langle b_{\eta \hat{\ } \langle n \rangle} : n \in u_\eta \rangle$ is a Δ -system.

The rest of the proof should be easy, too. \square

2. NO MINIMAL ULTRAFILTER ON THE STANDARD SYSTEM

Theorem 2.1. *Assume that \mathbb{N}_* is an expansion of \mathbb{N} with countable vocabulary or \mathbb{N}_* is an ordinary model of PA_τ , for some countable $\tau \supseteq \tau_{PA}$ such that \mathbb{N}_* is countable. Then there is M such that*

- (a) $\mathbb{N}_* \prec M$;
- (b) $\|M\| = \aleph_1$;
- (c) $SSy(M)$, the standard system of M (see Definition 1.3.) has no minimal ultrafilter on it (see Definition 1.4.), moreover
- (d) there is no Q -point on $SSy(M)$;
- (e) $SSy(M)$ is arithmetically closed.

Proof. 2.1.

Stage A:

Without loss of generality \mathbb{N}_* is the Skolem Hull of \emptyset as we can expand it by \aleph_0 individual constants.

We shall choose a sentence $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})(\tau^*)$ with $\tau^* \supseteq \tau(\mathbb{N}_*)$ and prove that it has a model, and for every model M^+ of ψ , the model $M^+ \upharpoonright \tau(\mathbb{N}_*)$ is as required. By the completeness theorem for $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ it is enough to prove that ψ has a model in some forcing extension; of course it is crucial that ψ can be explicitly defined hence $\in \mathbf{V}$.

Stage B:

Recall $cd = cd_0 : \mathcal{H}(\aleph_0) \rightarrow \omega$ be one-to-one onto and definable in \mathbb{N} by a bounded formula in the natural sense (see 1.4).

Let $\mathbf{V}_0 = \mathbf{V}$ and $\lambda = (2^{\aleph_0})^+$.

Let $\mathbb{R}_0 = \text{Levy}(\aleph_1, 2^{\aleph_0})$, let $\mathbf{G}_0 \subseteq \mathbb{R}_0$ be generic over \mathbf{V}_0 and let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_0]$, i.e. in $\mathbf{V}_0^{\mathbb{R}_0}$ we have CH.

In \mathbf{V}_1 we have $\lambda = \aleph_2$ and let \mathbb{R}_1 be \mathbb{P}_{ω_2} , where

$$\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle \text{ is a CS iteration,}$$

each \mathbb{Q}_α is a Laver forcing; there are many other possibilities, let $\eta_\alpha \in {}^\omega \omega$ (increasing) be the $\mathbb{P}_{\alpha+1}$ -name of the \mathbb{Q}_α -generic real and $\nu_\alpha = \langle cd(\eta_\alpha \upharpoonright n) : n < \omega \rangle$. Let $\mathbf{G}_1 \subseteq \mathbb{R}_1$ be generic over \mathbf{V}_1 and $\mathbf{V}_2 = \mathbf{V}_1[\mathbf{G}_1]$ and let $\eta_\alpha = \check{\eta}_\alpha[\mathbf{G}_1]$, $\nu_\alpha = \langle cd(\eta_\alpha \upharpoonright n) : n < \omega \rangle = \check{\nu}_\alpha[\mathbf{G}_1]$.

Let D^2 be a non-principal ultrafilter on ω in the universe \mathbf{V}_2 .

\boxplus_1 In the universe \mathbf{V}_2 let $M_1 = \mathbb{N}_*^\omega / D^2$, let $a_\alpha = \eta_\alpha / D^2 \in M_1$

and note

\boxplus_2 $\text{SSy}(M_1) = \mathcal{P}(\mathbb{N})^{\mathbf{V}_2}$ hence is arithmetically closed

\boxplus_3 let $f_1 \in \mathbf{V}_2$ be the function from $\lambda = \omega_2^{\mathbf{V}_1} = \omega_2^{\mathbf{V}_2}$ into M_1 defined by $f_1(\alpha) = a_\alpha$.

Stage C:

In \mathbf{V}_1 (yes, not in \mathbf{V}_2) let the forcing notion $\mathbb{R}_2 := \mathbb{P}_{\omega_2}^+$ and the set K be defined as follows (so $\mathbf{B} \in \mathbf{V}_1$ below, which is equivalent to $\mathbf{B} \in \mathbf{V}_0$, similarly for u ; so in $\boxplus_4(\alpha), \underline{A}$ is a \mathbb{P}_{ω_2} -name):

\boxplus_4 (α) $K := \{(\alpha, u, \underline{A}) : u \subseteq \lambda \text{ is countable, } \alpha \in u, \underline{A} = \mathbf{B}(\dots, \eta_\beta, \dots)_{\beta \in u},$

\mathbf{B} a Borel function from ${}^{\text{otp}(u)}(\omega^\omega)$ to $\mathcal{P}(\omega)$ such that.

$\Vdash_{\mathbb{P}_{\omega_2}}$ “ $\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))$ has $\leq \eta_\alpha(n)$ members,

moreover

$0 = \lim_n (|\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))| / \eta_\alpha(n)$ ”};

(β) $\mathbf{p} \in \mathbb{P}_{\omega_2}^+$ iff:

(a) $\mathbf{p} = (p, h) = (p_{\mathbf{p}}, h_{\mathbf{p}})$;

(b) $p \in \mathbb{P}_{\omega_2}$;

(c) h a function from some finite subset $K_{\mathbf{p}}$ of K to ω_1 ;

(d) if $(\alpha_\ell, u_\ell, \underline{A}_\ell) \in K_{\mathbf{p}}$ for $\ell = 1, 2$ and $h(\alpha_1, u_1, \underline{A}_1) = h(\alpha_2, u_2, \underline{A}_2)$ and $u_1 \subseteq \alpha_2$ then $p \Vdash_{\mathbb{P}_{\omega_2}}$ “ $\underline{A}_1 \cap \underline{A}_2$ is finite”;

(γ) $\mathbb{P}_{\omega_2}^+ \models \mathbf{p} \leq \mathbf{q}$ iff:

(a) $\mathbb{P}_{\omega_2} \models p_{\mathbf{p}} \leq p_{\mathbf{q}}$;

(b) $h_{\mathbf{p}} \subseteq h_{\mathbf{q}}$.

Now,

(*)₀ if $p \in \mathbb{P}_{\omega_2}$, $\alpha < \omega_2$ and $p \Vdash$ “ $\underline{A} \subseteq \omega$ satisfies $\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))$ has $\leq \eta_\alpha(n)$ members for every n large enough and $0 = \lim \langle |\underline{A} \cap [\eta_\alpha(n+1), \eta_\alpha(n+2))| / \eta_\alpha(n) : n < \omega \rangle$ ” then we can find a triple (q, u, \underline{A}') such that:

(α) $\mathbb{P}_{\omega_2} \models “p \leq q”$;

(β) $\text{Dom}(q) = u$;

(γ) u a countable set of ordinals $< \lambda$ (in \mathbf{V}_1 equivalently in \mathbf{V}_0);

(δ) $q \Vdash “\underline{A} = \underline{A}'”$

(ϵ) $\underline{A}' = \mathbf{B}(\dots, \eta_{\alpha_i}, \dots)_{i < \text{otp}(u)}$ where α_i is the i -th member of u , for some Borel function \mathbf{B} from ${}^{\text{otp}(u)}(\omega^\omega)$ to $\mathcal{P}(\omega)$ so $\mathbf{B} \in \mathbf{V}_1$ equivalently \mathbf{V}_0 ;

(ζ) $q(\alpha_i) = \mathbf{B}_i(\dots, \eta_{\alpha_j}, \dots)_{j < i}$ for every $i < \text{otp}(u)$ for some Borel function \mathbf{B}_i from ${}^i(\omega^\omega)$ to Laver forcing, of course, \mathbf{B}_i is from \mathbf{V}_0 .

[Why? Standard proof.]

(*)₁ $\mathbb{P}_{\omega_2}^+$ satisfies the \aleph_2 -c.c.

[Why? We need a property of the iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ stated in Claim 1.8. In more detail, given a sequence $\langle \mathbf{p}_\alpha : \alpha < \omega_2 \rangle$ of members of $\mathbb{P}_{\omega_2}^+$, for each $\alpha < \omega_2$, let $\mathbf{p}_\alpha = (p_\alpha, h_\alpha)$; and without loss of generality for each $(\alpha_1^*, u_1^*, A_1^*) \in K_{\mathbf{p}_\alpha}$ for some u^1, A^1 , the tuple (p_α, u, A^1) is like (q, u, A') in $(*)_0, (\beta) - (\zeta)$ and $(\alpha, u, A) \in \text{Dom}(h_\alpha) \Rightarrow u \subseteq \text{Dom}(p_\alpha)$. Letting $u_\alpha = \text{Dom}(p_\alpha)$, we can find a stationary $S \subseteq \{\delta < \omega_2 : \text{cf}(\delta) = \aleph_1\}$ and $p_*, \gamma(*)$, such that:

- $u_\delta \cap \delta = u_*$ for $\delta \in S$ and $u_\alpha \subseteq \delta$ for $\alpha < \delta \in S$;
- $p_\delta \upharpoonright \delta \leq p_* \in \mathbb{P}_\delta$ for $\delta \in S$;
- without loss of generality $p_\delta \upharpoonright \delta = p_*$ for $\delta \in S$;
- $\text{otp}(u_\delta) = \gamma(*)$ for $\delta \in S$;
- if $\delta_1, \delta_2 \in S$ then the order preserving function $\text{OP}_{u_{\delta_2}, u_{\delta_1}}$ from u_{δ_1} onto u_{δ_2} maps \mathbf{p}_{δ_1} to \mathbf{p}_{δ_2} .

Let $\delta(*) = \text{Min}(S)$ and $\mathbf{G}_{\delta(*)}^1 \subseteq \mathbb{P}_{\delta(*)}$ be generic over \mathbf{V}_1 such that $p_* \in \mathbf{G}_{\delta(*)}^1$. Now we apply the conclusion of Claim 1.8 to $\mathbb{P}_{\omega_2}/\mathbf{G}_{\delta(*)}$, the rest should be clear.

For $\delta \in S$, let $\alpha_\delta = \text{otp}(u_\delta \setminus \delta_*)$, \mathbf{h}_δ be the order preserving function from α_δ onto $u_\delta \setminus \delta$ and $(p'_\delta, h'_\delta) \in \mathbb{P}_{\alpha_\delta}$ be such that \mathbf{h}_δ maps (p'_δ, h'_δ) to (p_δ, h_δ) .

Clearly $\alpha_\delta, p'_\delta, h'_\delta$ are the same for all $\delta \in S$, so call them $\alpha(*), p', h'$ and applying 1.8. with $p', (\{\alpha, A\} : \text{for some } u \text{ the tuple } (\alpha, u, A) \text{ belongs to } \text{Dom}(h))$ here stands for $p, \{(\alpha_k, \beta_k) : k < k(*)\}$ there and get p'_1, p'_2 as there.

Let $\delta_1 < \delta_2$ be from S , let q_{δ_1} be $\mathbf{h}_{\delta_1}(p'_1), q_{\delta_2}$ be $\mathbf{h}_{\delta_2}(p'_2)$. [Easily, $p_{\delta_\ell} \leq q_{\delta_\ell}$ and $q_{\delta_1} \cup q_{\delta_2}$ is a common upper bound of $p_{\delta_1}, p_{\delta_2}$ in $\mathbb{P}_{\omega_2}^+/\mathbf{G}_{\delta(*)}^1$.]

(*)₂ $\mathbb{P}_{\omega_2}^+$ collapses ω_1 to \aleph_0 .

[Why? Easy but we can also use $\mathbb{P}_{\omega_2}^+ \times \text{Levy}(\aleph_0, \aleph_1)$ instead of $\mathbb{P}_{\omega_2}^+$.]

(*)₃ the function $p \mapsto (p, \emptyset)$ is a complete embedding of \mathbb{P}_{ω_2} into $\mathbb{P}_{\omega_2}^+$.

[Why? Should be clear.]

Stage D: Let $\mathbf{G}_2 = \mathbf{G}_1^+ \subseteq \mathbb{P}_{\omega_2}^+$ be generic over $\mathbf{V}_1, \mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ and by $(*)_3$ without loss of generality $\mathbf{G}_1 = \{p : (p, h) \in \mathbf{G}_2\}$. So $\mathbf{V}_3 = \mathbf{V}_1[\mathbf{G}_2]$ is a generic extension of \mathbf{V}_2 and let $f_2 = \cup\{h : (p, h) \in \mathbf{G}_2\}$.

So,

(*)₄ in \mathbf{V}_3 if $f_2(\alpha_1, u_1, A_1) = f_2(\alpha_2, u_2, A_2)$ and $u_1 \subseteq \alpha_2$, then $A_1[\mathbf{G}_1] \cap A_2[\mathbf{G}_1]$ is finite.

In \mathbf{V}_3 let M_2 be an elementary submodel of $(\mathcal{H}(\mathfrak{M}_\omega), \in, \dots, \mathbf{V}_\ell \cap \mathcal{H}(\mathfrak{M}_\omega), \dots)_{\ell=0,1,2}$ of cardinality $\lambda = \aleph_1^{\mathbf{V}_3}$ which includes the sets $\{\alpha : \alpha \leq \lambda\} = \{\alpha : \alpha \leq \omega_1^{\mathbf{V}_3}\}, \{M_1, f_1, f_2, \mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2\}$ and (the universe of) M_1 , see end of stage B, note that $\|M_2\| \subseteq |M_2|$.

Let f_0 be a one-to-one function from M_1 onto M_2 , let M_3 be a model such that f_0 is an isomorphism from M_1 onto M_3 . Lastly, let M_4 be M_3 expanded by $c_0 =$

$\lambda = \omega_2^{\mathbf{V}_1} = \omega_1^{\mathbf{V}_3}, c_1^{M_4} = \omega_1^{\mathbf{V}_1}, c_2^{M_4} = M_1, d_{0,\ell}^{M_4} = \mathbf{G}_\ell, d_{1,\ell} = \mathbb{R}_\ell, d^{M_4} = \mathbb{N}_*, \langle d_{2,n}^{M_4} : n < \omega \rangle$
 list the members of \mathbb{N}_* , $\mathcal{Q}_0^{M_4} = |\mathbb{N}_*|, \in^{M_2} = \in^{\mathbf{V}_3} \upharpoonright |M_2|, F_0^M = f_0, F_1^{M_4} = f_0 \circ f_1$, see
 end of Stage B, $F_2^{M_4} = f_2, P_\ell^M = \mathbf{V}_\ell \cap M_2$ for $\ell = 0, 1, 2$ (so F_ℓ is a unary function
 symbol, P_ℓ is a unary predicate) and lastly $<_*^M$, a linear order of $|M_2| = |M_4|$ of
 order type $\omega_1^{\mathbf{V}_3}$.

We define the sentence ψ : it is the conjunction of the following countable sets
 and singletons of sentences of $\mathbb{L}_{\aleph_1, \aleph_0}(\mathbf{Q})$ in the vocabulary $\tau(M_4)$ such that $M^+ \models$
 ψ iff:

- (A) $M^+ \upharpoonright \tau(\mathbb{N}_*)$ is isomorphic to \mathbb{N}_* , of course, $M^+ \upharpoonright \tau(\mathbb{N}_*)$ has universe $\mathcal{Q}_0^{M^+}$;
- (B) M^+ is uncountable, moreover $M^+ \models (\mathbf{Q}x) (x \text{ an ordinal } < c_0)$;
- (C) $<_*^{M^+}$ is a linear order;
- (D) every proper initial segment by $<_*^{M^+}$ is countable;
- (E) $(|M^+|, \in^{M^+})$ is a model ZFC^- (even a model of $\text{Th}(\mathcal{H}(\beth_\omega)^{\mathbf{V}_3}, \in)$);
- (F) the function $F_1^{M^+} : \{a : M^+ \models \text{“}a \text{ an ordinal } < c_0\text{”}\} \rightarrow M^+$ is one-to-one;
- (G) $M^+ \models \text{“}K \text{ is as above”}$;
- (H) $F_2^{M^+} : K^{M^+} \rightarrow \{a : M \models \text{“}a \text{ an ordinal } < c_1\text{”}\}$ is as above;
- (I) $M^+ \models \text{“for every } B \text{ we have } B \in \mathcal{P}(\mathbb{N}) \wedge P_2(B) \text{ iff } B = A \cap \mathbb{N} \text{ for some}$
 definable subset of A in the model c_2 ”.

It is easy to check that:

- (*)₅ $\psi \in \mathbf{V}_0$ and
- (*)₆ $M_4 \models \psi$ in \mathbf{V}_3 .

Hence as the completeness theorem for $\mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$ gives absoluteness

- (*)₇ ψ has a model in $\mathbf{V} = \mathbf{V}_0$ call it M_5 .

By renaming without loss of generality:

- (*)₈ (a) if $M_5 \models \text{“}a \text{ is the } n\text{-th natural number”}$ then $a = n$;
- (b) if $M_5 \models \text{“}A \subseteq \omega\text{”}$ then $A = \{n : M_5 \models \text{“}n \in A\text{”}\}$;
- (c) if $M_5 \models \text{“}b \in {}^\omega\omega\text{”}$ then $b = \{(n_1, n_2) : M_5 \models f(n_1) = n_2\}$;
- (*)₉ let $N'_* = M_5 \upharpoonright \tau(\mathbb{N}_*)$, so isomorphic to N_* , let $N = M_5 \upharpoonright \{\in\}$
- (*)₁₀ (a) let M'_1 be $c_2^{M_5}$ naturally defined;
- (b) so $M = M'_1$ is a model of $\text{Th}(N'_*) = \text{Th}(N_*)$, $N'_* \prec M'_1$ and $\|M'_1\| = \aleph_1$;
- (c) let \mathcal{A} be $\text{SSy}(M)$, the standard system of M ;

Clearly,

- (*)₁₁ (a) $N \models \text{“}ZC\text{”}$;
- (b) M is a model of $\text{Th}(\mathbb{N}_*)$ and $N_* \prec M$;
- (*)₁₂ let $\mathbb{R}'_\ell = d_{1,\ell}^{M_5}$ and $\mathbf{G}'_\ell = d_{2,\ell}^{M_5}$ and let $\mathbf{V}'_\ell = (P_\ell^{M_5}, \in^{M_5})$ for $\ell = 0, 1, 2$.

Stage E:

Clearly M is an uncountable elementary extension of \mathbb{N}_* , by clauses (A),(B) of Stage D and without loss of generality $\|M\| = \aleph_1$, so M satisfies clauses (a),(b) of Theorem 2.1. To prove clause (e) recall \boxplus_2 and clause (I) above hence $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed; this implies \mathcal{A} is a Boolean subalgebra. Also clause (d) implies clause (c), anyhow to prove them, assume toward contradiction that D is an ultrafilter on \mathcal{A} which is minimal or just a Q -point. Let $X = \{a : N \models \text{“}a \text{ is an ordinal } < \omega_1 \text{”}\}$, so X is really an uncountable set. For each $a \in X$ define a sequence $\rho_a \in {}^\omega\omega$ by $\rho_a(n) = k$ iff $M^+ \models \text{“}F_1(a)(n) = k\text{”}$.

Clearly ρ_a is an increasing sequence in ${}^\omega\omega$, hence by the assumption toward contradiction, there is $A_a \in D \subseteq \mathcal{A}$ such that $A_a \cap [\rho_a(n+1), \rho_a(n+2))$ has at most one element (or just $\leq \rho_a(n)$ elements) for each $n < \omega$.

So for some element \underline{A}_a of $N, N \models \text{“}\underline{A}_a, \text{ in } \mathbf{V}'_1, \text{ is a } \mathbb{R}_1\text{-name of a subset of } \omega \text{ and } \underline{A}_a[\mathbf{G}'_1] = A_a\text{”}$.

Clearly $M^+ \models \text{“for some countable subset } u \text{ of } \omega_2^{\mathbf{V}'_1} = \omega_1^{\mathbf{V}'_3} \text{ from } \mathbf{V}'_1 \text{ and Borel function } \mathbf{B} \text{ from } \mathbf{V}'_1 \text{ we have } A_a = \mathbf{B}_a(\dots, \rho_b, \dots)_{b \in u_a} \text{ (so some } p \in \mathbf{G}_2^+ \text{ forces } \underline{A}_a \text{ satisfies this)”}$. So using $F_2^{M^+}$ there are $a_1 \neq a_2$ from X such that the parallel of clause $(\beta)(d)$ of stage C holds, see clause (G) of stage D, so two members of D are almost disjoint, contradiction. \square

Remark 2.2. 1) Note that in 2.1 we can replace \mathbb{Q}_0 by any forcing notion similar enough, see [6].

2) We can strengthen 2.1 by replacing “ Q -point” by a weaker statement.

Similarly we can weaken the demands on how “thin” is \underline{B} in 1.8 and in the proof of 2.1.

Acknowledgments. The author would like to thank Shimoni Garti and the referee for helpful comments.

The author also thanks Alice Leonhardt for the beautiful typing. First version (F922) typed July 2008.

The author would like to thank the Israel Science Foundation (Grant No. 710/07) and the US-Israel Binational Science Foundation (Grant No. 2006108) and the National Science Foundation (Grant No. DMS 0600940) for partial support of this research. Publication 944.

REFERENCES

- [1] Ali Enayat, *A standard model of Peano arithmetic with no conservative elementary extension*, Annals of Pure and Applied Logic **56** (2008), 308–318.
- [2] H. Jerome Keisler, *Model theory for infinitary logic. Logic with countable conjunctions and finite quantifiers*, Studies in Logic and the Foundations of Mathematics, vol. 62, North-Holland Publishing Co., Amsterdam–London, 1971.
- [3] J. Knight and M. Nadel, *Models of arithmetic and closed ideals*, Journal of Symbolic Logic **47** (1982), 883–840.

- [4] Richard Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), 151–169.
- [5] Saharon Shelah, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.
- [6] Andrzej Roslanowski and Saharon Shelah, *Norms on possibilities I: forcing with trees and creatures*, Memoirs of the American Mathematical Society **141** (1999), no. 671, xii + 167, arxiv:math.LO/9807172.
- [7] Saharon Shelah, *Models of PA: when two elements are necessarily order automorphic*, Mathematical Logic Quarterly **61** (2015), 399–417, arxiv:1004.3342.
- [8] Ali Enayat and Saharon Shelah, *An improper arithmetically closed Borel subalgebra of $P(\omega)$ mod FIN*, Topology and its Applications **158** (2011), 2495–2502.
- [9] Saharon Shelah, *Models of expansions of \mathbb{N} with no end extensions*, Mathematical Logic Quarterly **57** (2011), 341–365, arxiv:0808.2960.

(Received: May 9, 2017)
(Revised: January 8, 2018)

Saharon Shelah
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
The Hebrew University of Jerusalem
Jerusalem, 91904, Israel
and
Department of Mathematics
Hill Center - Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Roads
Piscataway, NJ 08854-8019 USA
shelah@math.huji.ac.il
URL: <http://shelah.logic.at>

